COMPLEX NUMBERS

Introduction: Euler was the first mathematician to introduce the symbol i (iota) for the square root of – 1 with the property $i^2 = -1$. He also called this symbol as the imaginary unit.

The symbol i: It is customary to denote the complex number (0, 1) by i (iota). With this notation $j^2 = (0, 1)(0, 1) = (0.\ 0 - 1.\ 1,\ 0.\ 1 + 1.0) = (-1, 0)$ so that i may be regarded as the square root of – 1. Using the symbol i, we may write the complex number (a, b) as a + ib. For, we have $a + ib = (a, 0) + (0, 1)(b, 0) = (a, 0) = (a, 0) + i(0, b) = (a, 0) + (0.\ b - 1.\ 0,\ 0.\ 0 + 1.\ b) = (a, 0) + (0, b) = (a + 0, 0 + b) = (a, b)$.

To find the values of i^n, $n > 4$, we first divide n by 4. Let m be the quotient and r be the remainder. Then $n = 4m + r$, where $0 \le r \le 3$. $\therefore\ i^n = i^{4m+r} = (i^4)^m\, i^r = (1)^m\, i^r = i^r$ $\qquad [\because\ i^4 = 1]$

Thus, if n > 4, then $i^n = i^r$, where r is the remainder when n is divided by 4.
The values of the negative integral powers of i are found as given below:

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i,\ i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1.$$

Complex Number: An ordered pair of real numbers a, b, written as (a, b) is called a complex number If we write z = (a, b), then a is called the real part of z and b the imaginary part of z. It is customary to write Re (z) = a, Im (z) = b, where Re. (z) stands for the real part of z and Im (z) stands for the imaginary part of z.

Equality of Complex Numbers: Two complex numbers $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ are equal if $a_1 = a_2$ and $b_1 = b_2$. Thus, $z_1 = z_2 \Rightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

Algebra of Complex Numbers: Let $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ Then we have the following definitions of the addition, subtraction, multiplication and division of z_1 and z_2.

Addition of Complex Numbers: Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$.
It follows from the definition that the sum $z_1 + z_2$ is a complex number such that

$$\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$$

and

$$\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$$

e.g. *If $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$, then $z_1 + z_2 = (2 + 3) + (3 - 2)i = 5 + i$*

Properties of Addition of Complex Numbers

(I) **Addition is Commutative:** *For any two complex numbers z_1 and z_2, we have $z_1 + z_2 = z_2 + z_1$.*

(II) **Addition is Associative:** *For any three complex numbers z_1, z_2, z_3, we have $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.*

(III) **Existence of Additive Identity:** *The complex numbers 0 = 0 + i 0 is the identity element for addition i.e. $z + 0 = z = 0 + z$ for all $z \in C$.*

(IV) **Existence of Additive Inverse:** *The complex number z = – z is the identity element for addition i.e. $z + (-z) = 0 = (-z) + z$ for all $z \in C$.*

Subtraction of Complex Numbers: Let $z_1 = a_1 + i\, b_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.

Thus, $z_1 - z_2 = z_1 + (-z_2) = (a_1 + ib_1) + (-a_2 - ib_2)$

$= (a_1 - a_2) + i\,(b_1 - b_2)$

Multiplication of complex numbers: Let $z_1 = a_1 + i\, b_1$ and $z_2 = a_2 + i\, b_2$ be two complex numbers. Then multiplication of z_1 with z_2 is defined as complex numbers $(a_1 a_2 - b_1 b_2) + i\,(a_1 a_2 - a_1 b_1)$.

Thus, $z_1 z_2 = (a_1 + ib_1)(a_2 + i\, b_2)$

$= (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + a_2 b_1)$

$\Rightarrow z_1 z_2 = [Re(z_1)\, Re(z_2) - Im(z_1)\, Im(z_2)] + i\,[Re(z_1)\, Im(z_2) - Re(z_2)\, Im(z_1)]$

e.g. *if* $z_1 = 3 + 2i$ *and* $z_2 = 2 - 3i$, *then* $z_1z_2 = (3 + 2i)(2 - 3i) = (3 \times 2 - 2 \times (-3)) + i(3 \times -3 + 2 \times 2) = 12 - 5i$

__Properties of Multiplication__

(I) **Multiplication is Commutative:** *For any two complex numbers* z_1 *and* z_2, *we have* $z_1 z_2 = z_2 z_1$.

(II) **Multiplication is Associative:** *For any three complex numbers* z_1, z_2, z_3, *we have* $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

(III) **Existence of Identity Element for Multiplication:** *The complex number* $I = 1 + i0$ *is the identity element for multiplication i.e. for every complex number z, we have* $z \cdot I = z = I \cdot z$.

(IV) **Existence of Multiplicative Inverse:** *Corresponding to every non-zero complex number* $z = a + ib$ *there exists a complex number* $z_1 = x + iy$ *such that*

$z \,.\, z_1 = 1 = z_1 \,.\, z.$

e.g. *Find the multiplicative inverse of* $z = 3 - 2i$.

Sol. *Using the above formula, we have*

$$z^{-1} = \frac{1}{3-2i} * \frac{3+2i}{3+2i} \Rightarrow \frac{3+2i}{3^2-(2i)^2}$$

$$\Rightarrow \frac{3+2i}{9-(-4)} = \frac{3}{13} + \frac{2i}{13}$$

$$\Rightarrow \quad z^{-1} = \frac{3}{3^2+(-2)^2} + \frac{i(-(-2))}{3^2+(-2)^2} = \frac{3}{13} + \frac{2}{13} i$$

Division: The division of a complex number z_1 by a non-zero complex number z_2 is defined as the product of z_1 by the multiplicative inverse of z_2 Thus,

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then

$$\frac{z_1}{z_2} = (a_1 + ib_1)\left(\frac{a_2}{a_2^{\,2}+b_2^{\,2}} + \frac{i(-b_2)}{a_2^{\,2}+b_2^{\,2}}\right) = \frac{a_1a_2 + b_1b_2}{a_2^{\,2}+b_2^{\,2}} + i\frac{(a_2b_1 - a_1b_2)}{a_2^{\,2}+b_2^{\,2}}$$

e.g. *If* $z_1 = 2 + 3\, i$ *and* $z_2 = 1 + 2\, i$, *then*

$$\frac{z_1}{z_2} = z_1 \,.\, \frac{1}{z_2} = (2 + 3\, i)\left(\frac{1}{1+2i}\right) = (2 + 3\, i)\left(\frac{1}{5} - \frac{2}{5}i\right)$$

$$= \left(\frac{2}{5} + \frac{6}{5}\right) + i\left(-\frac{4}{5} + \frac{3}{5}\right) = \frac{8}{5} - \frac{1}{5}i$$

Multiplicative Inverse or Reciprocal of a Complex Number:

Let z = a + ib = (a, b) be a complex number

Since 1 + i0 is the multiplicative identity, therefore if x + iy is the multiplicative inverse of z = a + ib.

Then (a + ib) (x + iy) = 1 + i0

(ax – by) + i (ay + bx) = 1 + i0

ax – by = 1 bx + ay = 0

$x = \frac{a}{a^2 + b^2}$, $y = \frac{-b}{a^2 + b^2}$, if $a^2 + b^2 \neq 0$ i.e. $z \neq 0$

Thus, the multiplicative inverse of a + ib is $\frac{a}{a^2 + b^2} + \frac{i(-b)}{a^2 + b^2}$. The multiplicative inverse of z is denoted by 1/z or z^{-1}.

Modulus of a Complex Number: The modulus of a complex number z = a + i b is denoted by | z | and is defined as

$$|Z| = \sqrt{a^2 + b^2} = \sqrt{\{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2}.$$

Clearly, $|Z| \geq 0$ for all $z \in C$.

e.g. If $z_1 = 3 - 4i$, $z_2 = -5 + 2i$, $z_3 = 1 + \sqrt{-3}$, then

$|Z_1| = \sqrt{3^2 + (-4)^2} = 5$, $|z_2| = \sqrt{(-5)^2 + 2^2} = \sqrt{29}$

and $|Z_3| = |1 + i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = 2$.

Properties of Moduli

(1) $|z| = |\bar{z}|$

(2) $|z_1 z_2| = |z_1||z_2|$

(3) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(4) $|z_1 + z_2| \leq |z_1| + |z_2|$

(5) $|z_1 - z_2| \geq |z_1| - |z_2|$

(6) $|z_1 - z_2| \leq |z_1| + |z_2|$

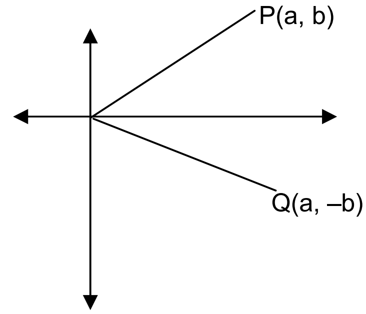
(7) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2)$ or

$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

(8) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1, \bar{z}_2)$ or $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \cos(\theta_1 - \theta_2)$,

(9) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$.

Conjugate of a Complex Number: If z = a + ib is a complex number, then a – ib is called the conjugate of z and is denoted by $\bar{z}$



By definition if z = (a, b), then $\bar{z}$ = (a, – b)
Thus if (a, b) represents z, in real axis If
(r, q) are polar coordinates of z, then the polar
coordinated of $\bar{z}$ are (r, – q) so that we having
arg (z) = – arg ($\bar{z}$).
In general, arg ($\bar{z}$) = 2π – arg (z)

Properties of Conjugate:

(1) $\left(\bar{\bar{z}}\right) = z$

(2) z = $\bar{z}$ < -- > z is purely real

(3) z = $-\bar{z} \Leftrightarrow z$ is purely imaginary

(4) z + $\bar{z}$ = 2Re (z)

(5) z – $\bar{z}$ = 2i Im (z)

(6) $z\bar{z} = |z|^2$

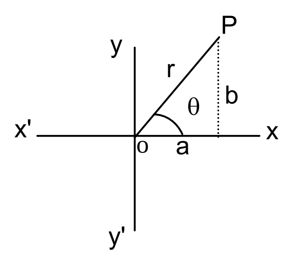
(7) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

(8) $\left(\overline{z_1 z_2}\right) = \overline{z_1}\ \overline{z_2}$

(9) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Geometrical Representation of Complex Number: The complex number z = a + ib = (a, b) is represented by a point P whose coordinates are (a, b) referred to rectangular axes XOX' and YOY', which are called real and imaginary axes respectively. Thus a complex number z is represented by a point P in a plane, and corresponding to every point in this plane there exists a complex number Such a plane is called Argand plane or Argand diagram or Complex plane or Gaussian plane.

Polar Form of a Complex Number: If P is a point (a, b) on the Complex plane corresponding to the complex number z = a + ib = (a, b), then from the figure, we have a = r cos θ, b = r sin θ where r = $+\sqrt{a^2 + b^2}$ = OP.



OP is the complex number z = a + ib. The length OP = $+\sqrt{a^2 + b^2}$ is called the modulus of z and is denoted by $|z|$. Thus, $|z| = +\sqrt{a^2 + b^2} = \sqrt{(\text{Re}(x))^2 + (\text{Im}(z))^2}$.

We have, $\tan\theta = \frac{b}{a} = \frac{\text{Im}(z)}{\text{Re}(z)} \therefore \theta = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$

θ is called the **amplitude or argument** of z and is written as θ = amp (z) or θ = arg (z) A value of θ satisfying – π < θ ≤ π is called the principal value of the argument. Thus, if r is the modulus and θ is the argument of a complex number z = a + ib, then z = r (cos θ + i sin θ).
(Because a = r cos θ, b = r sin θ). This is called the polar form of z.

Properties of Argument:

(1) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ (2) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Argument or Amplitude of a complex number z = x + iy for different signs of x and y

(I) **Argument of z = x + iy, when x > 0 and y > 0:** Since x and y both positive, therefore the point P(x, y) representing z = x + iy in the Argand plane lies in the first quadrant Let θ be the argument of z and let α be the acute angle satisfying $\tan\alpha = |y/x|$.

Thus, if x and y both are positive then the argument of z = x + iy is the acute angle (α) given by $\tan^{-1}|y/x|$.

(II) **Argument of z = x + iy, when x < 0 and y > 0:** Since x < 0 and y **>** 0 therefore the point p(x, y) representing z = x + iy in the Argand plane lies in the second quadrant Let θ be the argument of z and let a be the actue angle satisfying tan

$\alpha = |y/x|$ Then $\theta = \pi - \alpha$.

Thus, if x and y both are positive then the argument of z = x + iy is $\pi - \alpha$ where α is the acute angle given by $\tan^{-1}|y/x|$.

(III) **Argument of z = x + iy, when x < 0 and y < 0:** Since x > 0 and y < 0 therefore the point P(x, y) representing z = x + iy lies in the third quadrant Let θ be the argument of z and α be the acute angle given by $\tan\alpha = |y/x|$. Then we have $\theta = (\pi - \alpha) = \pi + \alpha$.

Thus, if x < 0 and y < 0 then argument of z = x + iy is $\pi - \alpha$ where α is the acute angle given by $\tan\alpha = |y/x|$

(IV) **Argument of z = x + iy, when x > 0 and y < 0.** Since x > 0 and y < 0, therefore the point P(x, y) representing z = x + iy lies in the fourth quadrant. Let θ be the argument of z and let α be the actue angle given by $\tan\alpha = |y/x|$ Then we have $\theta = -\alpha$.

Thus, if x > 0 and y > 0, then the argument of z = x + iy is $\pi - \alpha$ where α is the acute angle given by $\tan\alpha\, |y/x|$.

Algorithm for finding Argument of z = x + iy

<u>STEP I:</u> Find the value of $\tan^{-1}$ y/x lying between 0 and $\pi/2$. Let it be α

<u>STEP II.</u> Determine in which quadrant the point P(x, y) belong

If P(x, y) belongs to the first quadrant, then arg (z) = α

If P(x, y) belongs to the second quadrant, then arg (z) = $\pi - \alpha$

If P(x, y) belongs to the third quadrant, the arg (z) = $-(\pi - \alpha)$ or $\pi + \alpha$

If P(x, y) belongs to the fourth quadrant, then arg (z) $-\alpha$ or $2\pi - \alpha$

Some Standard Locii in the Argand Plane

1. If z is a vartiable point in the Argand plane such that arg (z) = θ, then locus of z is a straight line (excluding origin) through the origin inclined at an angle θ with x –axis
2. If z is a vairable point and z_1 is a fixed point in the Argand plane such that arg $(z - z_1) = \theta$ then locus of z is a straight line passing through the point representing z_1 and inclined at an angle θ with x-axis Note that the point z_1 is excluded form the locus.

3. If z is a variable point and z_1, z_2 are two fixed points in the Argand plane, then

∗ $|z - z_1| = |z - z_2| \Rightarrow$ locus of z is the perpendicular bisector of the line segment joining z_1 and z_2

∗ $|z - z_1| + |z - z_2|$ = const . $(\neq |z_1 - z_2|) \Rightarrow$ locus of z is an ellipse.

∗ $|z - z_1| + |z - z_2| = |z_1 - z_2| \Rightarrow$ locus of z is the line segment joining z_1 and z_2

∗ $|z - z_1| - |z - z_2| = |z_1 - z_2| \Rightarrow$ locus of z is a straight line joining z_1 and z_2 but does not lie between z_1 and z_2

∗ $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2 \Rightarrow$ locus of z is circle with z_1 and z_2 as the extremities of diameter.

∗ $|z - z_1| = k|z - z_2|, k \neq 1 \Rightarrow$ locus of z is a segment of circle.

∗ $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ (fixed) $\Rightarrow$ locus of z is a segment of circle

∗ $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pi/2 \Rightarrow$ Locus of z is a circle with z_1 and z_2 as the vertices of diameter.

∗ $\arg\left(\frac{z - z_1}{z - z_2}\right) = 0$ or $\pi \Rightarrow$ Locus of z is a straight line passing through z_1 and z_2

Complex Form of Various Locii:

1. The equation of a circle with centre at z_1 and result r is $|z - z_1| = r$
2. $|z - z_1| < r$ represents interior of a circle $|z - z_1| = r$ and $|z - z_1| > r$ represents the exterior of the circle $|z - z_1| = r$
3. $\left|\frac{z - z_1}{z - z_2}\right| = K$ represents a circle if $K \neq 1$. When K = 1, it represents a straight line.
4. $|z - z_1| + |z - z_2| = 2a$ where $a \in R^+$ represents an ellipse having foci at z_1 and z_2
5. Each complex cube root of unity is the square of the other.
6. Cube roots of –1 are –1, $-\omega$ and $-\omega^2$

Remark: The idea of finding cube roots of 1 and – 1 can be extended to find cube roots of any real number. If a is any positive real number then $a^{1/3}$ has values $a^{1/3}$, $a^{1/3}\omega$, $a^{1/3}\omega^2$. If a is a negative real number then $a^{1/3}$ has values – $|a|^{1/3}$ has values $-|a|^{1/3}$, $-|a|^{1/3}\omega$ and $-|a|^{1/3}\omega^2$ For example, $8^{1/3}$ has values 2, 2ω and $2\omega^2$ where as $(-8)^{1/3}$ has – 2, – 2 ω and $-2\omega^2$.

SQUARE ROOTS OF A COMPLEX NUMBER

Let a + ib be a complex number such that $\sqrt{a+ib}$ = x + iy, where x and y are real numbers.

Then, $\sqrt{a+ib} = x + iy \Rightarrow (a + ib) = (x + iy)^2$

$\Rightarrow a + ib = (x^2 - y^2) + 2\,i\,xy$

$\Rightarrow x^2 - y^2 = a$ ……….(i) and, $2\,xy = b$ …..(ii) [On equating real and imaginary parts.]

Now, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4\,x^2\,y^2$

$\Rightarrow (x^2 + y^2)^2 = a^2 + b^2 \Rightarrow (x^2 + y^2) = \sqrt{a^2+b^2}$ ……(iii)

Solving the equations (i) and (ii), we get

$x^2 = (1/2)\ \{\sqrt{a^2+b^2} + a\}$ and $y^2 = (1/2)\{\sqrt{a^2+b^2} - a\}$

$\Rightarrow x = \pm\sqrt{\left(\frac{1}{2}\right)\{\sqrt{a^2+b^2}+a\}}$ and $y = \pm\sqrt{\left(\frac{1}{2}\right)\{\sqrt{a^2+b^2}-a\}}$

If b is positive, then by equation (ii), x and y are of the same sign. Hence,

$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{1}{2}\{a^2+b^2}+a\}+i\sqrt{\frac{1}{2}\{\sqrt{a^2+b^2}-a\}}\right\}$$

If b is negative, then by equation (ii), x and y are of different signs. Hence,

$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{1}{2}\{a^2+b^2}+a\}-i\sqrt{\frac{1}{2}\{\sqrt{a^2+b^2}-a\}}\right\}$$

E.g. *Find the square root of 7 – 24i*

Sol. Let $\sqrt{7-24i} = x + i\,y$. Then,

$\sqrt{7-24i} = x + iy \Rightarrow 7 - 24\,i = (x + iy)^2$

$\Rightarrow\quad 7 - 24\,i = (x^2 - y^2) + 2\,i\,xy$

$\Rightarrow\quad x^2 - y^2 = 7$ …..(i) & $2\,xy = -24$ ….(ii) [On equating real and imag. parts]

Now, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4\,x^2\,y^2$

$\Rightarrow\quad (x^2 + y^2)^2 = 49 + 576 = 625$

$\Rightarrow\quad x^2 + y^2 = 25$ …..(iii) $[\because x^2 + y^2 > 0]$

On solving (i) and (iii), we get

$x^2 = 16$ and $y^2 = 9 \Rightarrow x = \pm 4$ and $y = \pm 3$

From (ii), $2xy$ is negative. So, x and y are opposite signs.

$\therefore$ ($x = 4$ and $y = -3$) or ($x = -4$ and $y = 3$)

Hence, $\sqrt{7-24i} = \pm(4 - 3i)$

De–Moivere's Theorem

Statement: (i) If n $\in$ Z (the set of integers), then

$(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\sin n\,\theta$

(ii) If n $\in$ Q (the set of rational numbers), then $\cos n\,\theta + i\sin n\,\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Proof: (i) We divide the proof into the following three cases:

Case 1 *When n = 0.*

In this case $(\cos\theta + i\sin\theta)^n$

$= (\cos\theta + i\sin\theta)^0 = 1 = \cos(0.\theta) + i\sin(0.\theta)$

$= \cos n\,\theta + i\sin n\,\theta$ $[\because\ n = 0]$

So, the theorem holds good for $n = 0$.

Case II *When $n \in N$ **i.e.** n is a positive integer*

In this case we shall prove the theorem by induction on n.

the theorem is true for $n = 1$, because

$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta = \cos(1.\theta) + i\sin(1.\theta)$

Let us assume that the theorem is true for $n = m$. Then,

$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$ ….(a)

Now, we shall show that the theorem is true for $n = m + 1$.

We have, $(\cos\theta + i\sin\theta)^{m+1}$

$(\cos\theta + i\sin\theta)^m(\cos\theta + i\sin\theta)$

$(\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$ [Using (a)]

$(\cos m\theta\cos\theta - \sin m\theta\sin\theta) + i(\cos m\theta\sin\theta + \sin m\theta\cos\theta)$

$\cos(m\theta + \theta) + i\sin(m\theta + \theta)$

$\cos(m + 1)\theta + i\sin(m + 1)\theta$

So, the theorem holds good for $n = m + 1$.

Hence, by induction the theorem holds good for all $n \in N$.

Case III *When n is a negative integer.*

Let $n = -m$, where $m \in N$. Then,

$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \times \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i\sin m\theta$$

$= \cos(-m)\theta + i\sin(-m)\theta$

$= \cos n\theta + i\sin n\theta$ $[\because -m = n]$

Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in Z$.

(ii) Let $n = \frac{p}{q}$, where p, q are integers and $q > 0$. From part (i) we have

$$\left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)^q = \cos\left(\left(\frac{p\theta}{q}\right)q\right) + i\sin\left(\left(\frac{p\theta}{q}\right)q\right)$$

$= \cos p\theta + i\sin p\theta$

$\Rightarrow \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the values of $(\cos p\theta + i\sin p\theta)^{1/q}$

$\Rightarrow \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the values of $[(\cos\theta + i\sin\theta)^p]^{1/q}$

$\Rightarrow \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$

Remark 1 *The theorem is also true for $(\cos\theta - i\sin\theta)$ i.e.*

$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$, *because*

$(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$

$= \cos(n(-\theta)) + i\sin(n(-\theta))$

$= \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

Remark 2 $\dfrac{1}{\cos\theta + i\sin\theta}$ = $(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

Note 1: $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$

Note 2: $(\sin\theta + i\cos\theta)^n = [\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)]^n$

$= \cos(n\pi/2 - n\theta) + i\sin(n\pi/2 - n\theta)$

Note 3: $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

e.g. *If* $x_n = \cos\left(\dfrac{\pi}{2^n}\right) + i\sin\left(\dfrac{\pi}{2^n}\right)$ *prove that*

$x_1 . x_2 . x_3$ *….to infinity = –1.*

$$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}\right) \ldots.. \infty$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots.\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots.\right)$$

$$= \cos\left(\frac{\pi/2}{1-1/2}\right) + i\sin\left(\frac{\pi/2}{1-1/2}\right) \left[a + ar + ar^2 + \ldots \frac{a}{1-r}\right]$$

$= \cos\pi + i\sin\pi = -1.$

Roots of a Complex Number: Let $z = a + ib$ be a complex number, and let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then by De Moivre's theorem $r^{1/n}\left\{\cos\left(\dfrac{\theta}{n}\right) + i\sin\left(\dfrac{\theta}{n}\right)\right\}$ is one of the values of $z^{1/n}$. Here we shall show that $z^{1/n}$ has n distinct values.

We know that

$\cos\theta + i\sin\theta = \cos(2m\pi + \theta) + i\sin(2m\pi + \theta)$, $m = 0, 1, 2, \ldots\ldots..$

So, $z^{1/n} = r^{1/n}[\cos(2m\pi + \theta) + i\sin(2m\pi + \theta)]^{1/n}$

$$= r^{1/n}\left[\cos\frac{2m\pi + \theta}{n} + i\sin\frac{2m\pi + \theta}{n}\right]$$

Now by giving m the values 0, 1, 2,…, $(n-1)$; we shall obtain distinct values of $z^{1/n}$. For the values $m = n$, $n + 1$, …, the values of $z^{1/n}$ will repeat. For example, if $m = n$, then

$$z^{1/n} = r^{1/n}\left[\cos\frac{2n\pi + \theta}{n} + i\sin\frac{2n\pi + \theta}{n}\right]$$

$$= r^{1/n}\left[\cos\left(2\pi + \frac{\theta}{n}\right) + i\sin\left(2\pi + \frac{\theta}{n}\right)\right] \quad = r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

Which is same as the value obtained by taking $m = 0$. On taking $m = n + 1$, the values comes out to be identical with corresponding to $m = 1$. Hence $z^{1/n}$ has n distinct values.

Euler's form for complex numbers: Any complex number Z can be represented as

$$Z = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\overline{Z} = e^{-i\theta} = \cos\theta - i\sin\theta.$$

Cube Roots of Unity: Let $z = 1^{1/3} = 1(\cos 0 + i \sin 0)^{1/3} = (\cos 2r\pi + i \sin 2r\pi)^{1/3}$

where r in an integer, $= \cos\frac{2r\pi}{3} + i\sin\frac{2r\pi}{3}$ where r = 0, 1, 2

$\therefore$ z = 1, $\cos\frac{2r\pi}{3} + i\sin\frac{2r\pi}{3}$, $\cos\frac{4r\pi}{3} + i\sin\frac{4r\pi}{3}$ or z = 1, $\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

Hence, three cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ or 1, ω, ω^2 where $\omega = \frac{-1+i\sqrt{3}}{2}$

Properties of cube roots of unity:

(1) $1 + \omega + \omega^2 = 0$.

(2) Three cube roots of unity lie on a circle $|z| = 1$ and divide it in three equal parts.

(3) Cube roots of unity are in GP.

nth Roots of Unity: Let $z = 1^{1/n}$ Then $z = (\cos 0 + i \sin 0)^{1/n}$ (Because 1 = cos o + i sin 0)

$z = (\cos 2k\pi + i\sin 2r\pi)^{1/n}$ where $r \in Z$ (the set of integers.)

$z = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$ (Using De 'Moivre's theorem), where r = 0, 1, 2,…,(n – 1)

$z = e^{i\frac{2r\pi}{n}}$ where r = 0 , 1 , 2 ,…,(n – 1)

Let $\alpha = e^{i\frac{2\pi}{n}}$ Then nth roots of unity are 1, α, α^2,…, α^{n-1}

Properties of nth roos of unity:

(1) The n nth roots of unity in G.P.

(2) The product of n, nth roots of unity is $(-1)^{n-1}$.

(3) The n nth roots of unity lie on a circle $|z| = 1$.

Algorithm for finding n^th roots of a given Complex Number

Step 1: Write the given complex number in polar form.

Step 2: Add 2 m π to the argument.

Step 3: Apply De Moivre's theorem.

Step 4: Put m = 0, 1, 2, …, (n – 1) i.e. one less than the number in the denominator of the given index in the lowest form.

E.g. Find $\sqrt[3]{-1}$

Sol. Let $z = -1$. Then $|Z| = 1$ and arg $(z) = \pi$. So, the polar form of z is $(\cos\pi + i\sin\pi)$.

Now, $z^{1/3} = (\cos\pi + i\sin\pi)^{1/3}$

$= [\cos(2m\pi + \pi) + i\sin(2m\pi + \pi)]^{1/3}$

$= \cos(2m+1)\frac{\pi}{3} + i\sin(2m+1)\frac{\pi}{3}$, $m = 0, 1, 2$

For $m = 0$, $z^{1/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2}(1 + i\sqrt{3})$

For $m = 1$, $z^{1/3} = \cos\pi + i\sin\pi = -1$

From $m = 2$,

$z^{1/3} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right)$

$= \cos\frac{\pi}{3} - \sin\frac{\pi}{3} = \frac{1}{2}(1 - i\sqrt{3})$

Hence, the values of $z^{1/3}$ are $\frac{1}{2}(1 \pm i\sqrt{3})$, – 1.

LOGARITHM OF A COMPLEX NUMBER

Let $x + iy$ and $a + ib$ be two complex numbers such that $a + ib = e^{x+iy}$, then $x + iy$ is called logarithm of $a + ib$ to the base e and we write

$x + iy = \log_e(a + ib)$.

Since $e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi = 1$

$\therefore\ e^{x+iy} = e^{x+iy}e^{i2n\pi} = e^{x+i(2n\pi+y)}$ for all $n \in Z$

Hence, if $x + iy$ be logarithm of $a + ib$, then $x + i(2n\pi + y)$ is also a logarithm of $a + ib$ for all integral values of n. The value $x + i(y + 2n\pi)$ is called the general value of log $(a + ib)$ and is denoted by Log $(a + ib)$. Thus,

$$\text{Log}(a + ib) = 2n\pi i + \log(a + ib)$$

Let $a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$,

where $r = |Z|$ and $\theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$

Then

$\log(a + ib) = \log(re^{i\theta}) = \log r + i\theta$

$= \log\sqrt{a^2 + b^2} + i\tan^{-1}\left(\frac{b}{a}\right)$

or $\log(z) = \log|Z| + i\,\text{amp}(z)$, where $z = a + ib$.

e.g. $\log(1 + i) = \log|1 + i| + i\arg(1 + i)$

$= \log\sqrt{2} + i\pi/4 = \frac{1}{2}\log 2 + \frac{i\pi}{4}$

e.g. *Find* $(i)^i$

Sol. Let $A = (i)^i$. Then,

$\log A = \log(i)^i = i\log i = i\log(0 + i)$

$= i\{\log 1 = i\pi/2\}$ [$\because |i| = 1$ and $\arg(i) = \pi/2$]

$= i\{0 + i\pi/2\} = -\pi/2$

$A = e^{-\pi/2}$.